

GENERALIZED CONTINUOUS-TIME RANDOM WALKS (CTRW), SUBORDINATION BY HITTING TIMES AND FRACTIONAL DYNAMICS

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Abstract

Functional limit theorem for continuous-time random walks (CTRW) are found in general case of dependent waiting times and jump sizes that are also position dependent. The limiting anomalous diffusion is described in terms of fractional dynamics. Probabilistic interpretation of generalized fractional evolution is given in terms of the random time change (subordination) by means of hitting times processes.

Key words. Fractional stable distributions, anomalous diffusion, fractional derivatives, limit theorems, continuous time random walks, time change, Lévy subordinators, hitting time processes.

Running Head: Limit distributions for CTRW.

1 Introduction

Suppose $(X_1, T_1), (X_2, T_2), \dots$ is a sequence of i.i.d. pairs of random variables such that $X_i \in \mathbf{R}^d$, $T_i \in \mathbf{R}_+$ (jump sizes and waiting times between the jumps), the distribution of each (X_i, T_i) being given by a probability measure $\psi(dx dt)$ on $\mathbf{R}^d \times \mathbf{R}_+$. Let

$$N_t = \max\{n : \sum_{i=1}^n T_i \leq t\}.$$

The process

$$S_{N_t} = X_1 + X_2 + \dots + X_{N_t} \tag{1}$$

is called the continuous time random walk (CTRW) arising from ψ . These CTRW were introduced in [17] and found numerous applications in physics and economics (see e.g. [23], [14], [3], [12], [15] and references therein). Of particular interest are the situations,

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where T_i belong to the domain of attraction of a $\beta \in (0, 1)$ -stable law and X_i belong to the domain of attraction of a $\alpha \in (0, 2)$ -stable law. The limit distributions of appropriately normalized sums S_{N_i} were first studied in [7] in case of independent T_i and X_i (see also [11]). In [5] the rate of convergence in double array schemes was analyzed and in [14] the corresponding functional limit was obtained, which was shown to be specified by a fractional differential equations. The importance of the analysis of the case of dependent T_i and X_i was stressed both in [7] and [14]. Here we address this issue. Moreover we extend the theory to include possible dependence of (T_n, X_n) on the current position. Our method is quite different from those used in [7], [11], [14]. It is based on the finite difference approximations to continuous-time operator semigroups and applies the previous results of the author from [8] on stable-like processes.

It was noted in [14] that fractional evolution appears from the subordination of Levy processes by the hitting times of stable Levy subordinators. Implicitly this idea was present already in [19]. We are going to develop here the general theory of subordination of Markov processes by the hitting time process showing that this procedure leads naturally to (generalized) fractional evolutions. In particular, in spite of the remark from [14] that the method from [19] (going actually back to [17]) "does not identify the limit process" we shall give a rigorous probabilistic interpretation of the intuitively appealing (but rather formal) calculations from [19].

In the next Section we demonstrate our approach to the limits of CTRW by obtaining simple (but nevertheless seemingly new) limit theorems for position depending random walks with jump sizes from the domain of attraction of stable laws. In Section 3 these results will be extended to double scaled random walks, which are needed for the analysis of CTRW. Section 4 (which is independent of Section 2 and seems to be of independent interest) is devoted to the theory of subordination by hitting times. In Section 5 we combine the two bits of the theory from Sections 3 and 4 giving our main results on CTRW.

Let us fix some (rather standard) notations to be used throughout the paper. For a locally compact space X we denote by $C(X)$ the Banach space of bounded continuous functions (equipped with the the sup-norm) and by $C_\infty(X)$ its closed subspace consisting of functions vanishing at infinity. We denote by (f, μ) the usual pairing $\int f(x)\mu(dx)$ between functions and measures. By a continuous family of transition probabilities (CFTP) in X we mean as usual a family $p(x; dy)$ of probability measures on X depending continuously on $x \in X$, where probability measures are considered in their weak topology ($\mu_n \rightarrow \mu$ as $n \rightarrow \infty$ means that $(f, \mu_n) \rightarrow (f, \mu)$ as $n \rightarrow \infty$ for any $f \in C(X)$).

For a measure $\mu(dy)$ in \mathbf{R}^d and a positive number h we denote by $\mu(dy/h)$ the scaled measure defined via its action

$$\int g(z)\mu(dz/h) = \int g(hy)\mu(dy)$$

on functions $g \in C(\mathbf{R}^d)$.

The capital letters E and P are reserved to denote expectation and probability. The function $\delta(x)$ is the usual Dirac function (distribution).

2 Limit theorems for position dependent random walks

For a vector $y \in \mathbf{R}^d$ we shall always denote by \bar{y} its normalization $\bar{y} = y/|y|$, where $|y|$ means the usual Euclidean norm.

Fix an arbitrary $\alpha \in (0, 2)$. Let $S : \mathbf{R}^d \times S^{d-1} \mapsto \mathbf{R}_+$ be a continuous non-negative function that is symmetric with respect to the second variable, i.e. $S(x, y) = S(x, -y)$. It defines a family of α -stable d -dimensional symmetric random vectors (depending on $x \in \mathbf{R}^d$) specified by its characteristic function ϕ_x with

$$\ln \phi_x(p) = \int_0^\infty \int_{S^{d-1}} \left(e^{i(p, \xi)} - 1 - \frac{i(p, \xi)}{1 + \xi^2} \right) \frac{d|\xi|}{|\xi|^{1+\alpha}} S(x, \bar{\xi}) d_S \bar{\xi}, \quad (2)$$

where d_S denotes the Lebesgue measure on the sphere S^{d-1} . It is well known that it can be also rewritten in the form

$$\ln \phi_x(p) = C_\alpha \int_{S^{d-1}} |(p, \bar{\xi})|^\alpha S(x, \bar{\xi}) d_S \bar{\xi}$$

with a certain constant C_α .

Remark 1 *There are no obstacles for extending our theory to non-symmetric stable laws. But working with symmetric laws shorten the formulas essentially.*

Theorem 2.1 *Assume*

$$C_1 \leq \int_{S^{d-1}} |(\bar{p}, s)|^\alpha S(x, s) d_S s \leq C_2$$

for all p with some constants C_1, C_2 and that $S(x, s)$ has bounded derivatives with respect to x up to and inclusive order $q \geq 3$ (if $\alpha < 1$, the assumption $q \geq 2$ is sufficient). Then the pseudo-differential operator

$$Lf(x) = \ln \phi_x \left(\frac{1}{i} \frac{\partial}{\partial x} \right) f(x) = \int_0^\infty \int_{S^{d-1}} (f(x + y) - f(x)) \frac{d|y|}{|y|^{1+\alpha}} S(x, \bar{y}) d_S \bar{y} \quad (3)$$

generates a Feller semigroup T_t in $C_\infty(\mathbf{R}^d)$ with the space $C^{q-1}(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ being its invariant core.

This result is proven in [8] and [9].

Remark 2 *In [8] it is also shown that this semigroup has a continuous transition density (heat kernel), but we do not need it.*

Denote by $Z_x(t)$ the Feller process corresponding to the semigroup T_t . We are interested here in discrete approximations to T_t and $Z_x(t)$.

We shall start with the following technical result.

Proposition 2.1 *Assume that $p(x; dy)$ is a CFTP in \mathbf{R}^d from the normal domain of attraction of the stable law specified by (2). More precisely assume that for an arbitrary open $\Omega \in S^{d-1}$ with a boundary of Lebesgue measure zero*

$$\int_{|y|>n} \int_{\bar{y} \in \Omega} p(x; dy) \sim \frac{1}{\alpha n^\alpha} \int_\Omega S(x, s) d_S s, \quad n \rightarrow \infty, \quad (4)$$

(i.e. the ratio of the two sides of this formula tends to one as $n \rightarrow \infty$) uniformly in x . Assume also that $p(x, \{0\}) = 0$ for all x . Then

$$\min(1, |y|^2)p(x, dy/h)h^{-\alpha} \rightarrow \min(1, |y|^2)\frac{d|y|}{|y|^{\alpha+1}}S(x, \bar{y})d_S\bar{y}, \quad h \rightarrow 0, \quad (5)$$

where both sides are finite measures on $\mathbf{R}^d \setminus \{0\}$ and the convergence is in the weak sense and is uniform in $x \in \mathbf{R}^d$. If $\alpha < 1$, then also

$$\min(1, |y|)p(x, dy/h)h^{-\alpha} \rightarrow \min(1, |y|)\frac{d|y|}{|y|^{\alpha+1}}\int_{\Omega} S(x, \bar{y})d_S\bar{y}, \quad h \rightarrow 0,$$

holds in the same sense.

Remark 3 As the limiting measure has a density with respect to Lebesgue measure, the uniform weak convergence means simply that the measures of any open or closed set converge uniformly in x .

Proof. By (4)

$$\int_{|z|>A} \int_{\bar{z} \in \Omega} p(x; dz/h)h^{-\alpha} = \int_{|y|>A/h} \int_{\bar{y} \in \Omega} p(x; dy)h^{-\alpha} \sim \frac{1}{\alpha A^\alpha} \int_{\Omega} S(x, s)d_S s$$

as $h \rightarrow 0$. Hence

$$\int_{A<|z|<B} \int_{\bar{z} \in \Omega} p(x; dz/h)h^{-\alpha} \rightarrow \int_A^B \frac{d|z|}{|z|^{\alpha+1}} \int_{\Omega} S(x, s)d_S s.$$

Hence $p(x; dz/h)h^{-\alpha}$ converges weakly to $|z|^{-(\alpha+1)}d|z|S(x, z/|z|)d_S(z/|z|)$ on any set separated from the origin. It is easy to see that (5) follows now from the uniform bound

$$\int_{|y|<\epsilon} \min(1, |y|^2)p(x, dy/h)h^{-\alpha} \leq C\epsilon^{2-\alpha} \quad (6)$$

with a constant C . In order to prove (6) let us observe that

$$\int_{|y|>n} p(x, dy) \leq Cn^{-\alpha}$$

with a constant C uniformly for all x and $n > 0$ (in fact it holds for large enough n by 4 and is extended to all n , because all $p(x, dy)$ are probability measures). Hence for an arbitrary $\epsilon < 1$ one has

$$\int_{|y|<\epsilon} \min(1, |y|^2)p(x, dy/h)h^{-\alpha} = \int_{|z|<\epsilon/h} h^2|z|^2p(x, dy/h)h^{-\alpha}.$$

Representing this integral as the countable sum of the integrals over the regions

$$\epsilon/(2^{k+1}h) < y \leq \epsilon/(2^k h),$$

it can be estimated by

$$\sum_{k=0}^{\infty} h^2 \left(\frac{\epsilon}{2^k h} \right)^2 h^{-\alpha} C h^\alpha 2^{\alpha(k+1)} \epsilon^{-\alpha} = \sum_{k=0}^{\infty} C \epsilon^{2-\alpha} 2^\alpha 2^{-(2-\alpha)k}.$$

This yields (6), since the sum on the r.h.s. converges.

The improvement concerning the case $\alpha < 1$ is obtained similarly.

Consider the jump-type Markov process $Z^h(t)$ generated by

$$(L_h f)(x) = \frac{1}{h^\alpha} \int (f(x + hy) - f(x))p(x; dy) \quad (7)$$

For each h the operator L_h is bounded in $C_\infty(\mathbf{R}^d)$ and hence specifies a Feller semigroup there. The probabilistic interpretation of $Z^h(t)$ is as follows. Starting at a point x one waits a random $\theta = h^{-\alpha}$ -exponential time τ (i.e. distributed according to $P(\tau > t) = \exp(-t\theta)$) and then jumps to $x + hY$, where Y is distributed according to $p(x; dy)$. Then the same repeats starting from $x + hY$, etc. In case when p does not depend on x

$$Z^h(t) = h(Y_1 + \dots + Y_{N_t})$$

is a normalized random walk with the number of jumps N_t being a Poisson process with parameter $h^{-\alpha}$, so that $EN_t = th^{-\alpha}$. In particular, the number of jumps $n = N_t \sim th^{-\alpha}$ for small h so that $Z^h(1) \sim n^{-1/\alpha}(Y_1 + \dots + Y_n)$.

Theorem 2.2 *The semigroup T_t^h generated by L_h converges to the semigroup T_t generated by L . In particular, the corresponding processes converge in the sense of finite-dimensional marginal distributions.*

Remark 4 *Everywhere in this paper we work with the convergence of semigroups only. However by the standard results (see e.g. Theorem 19.25 in [6]) for Feller processes this convergence is equivalent to the convergence of the distributions of trajectories in an appropriate Skorokhod space of càdlàg paths.*

Proof. By (7)

$$(L_h f)(x) = \frac{1}{h^\alpha} \int (f(x + z) - f(x))p(x; dz/h),$$

and by Proposition 2.1 this converges to $Lf(x)$ as $h \rightarrow 0$ uniformly in x for $f \in C_\infty(\mathbf{R}^d) \cap C^2(\mathbf{R}^d)$. By a well known result (see e.g. [13]) the convergence of the generators on the core of the limiting semigroup implies the convergence of semigroups.

The next result concerns the approximations with a non-random number of jumps. Define the process $S_x^h(t) = S_x^h([t])$ (by the square bracket the integer part of a real number was denoted) via

$$S_x^h(0) = x, \quad S_x^h(1) = x + hY_1, \quad \dots, \quad S_x^h(j) = S_x^h(j-1) + hY_j, \dots$$

where each Y_j is distributed according to $p(S_{j-1}, dy)$. If $p(x; dy)$ does not depend on x , then

$$S_x^h(n) = x + h(Y_1 + \dots + Y_n)$$

is just a standard random walk.

We like to compare the Feller process $Z_x(t)$ on an arbitrary fixed time interval $[0, t_0]$ with the discrete approximations $S_x^h(t/\tau)$, when the number of jumps $n = t/\tau$ is connected with the scaling parameter h by $\tau = h^\alpha$.

Theorem 2.3 *Under the assumptions of Theorem 2.1 and Proposition 2.1 for any $f \in C_\infty(\mathbf{R}^d)$, $Ef(S_x^h(t/\tau))$ converges to $T_t f(x)$ uniformly on $t \in [0, t_0]$, as $\tau = h^\alpha \rightarrow 0$. In particular, the processes $S_x^h(t/\tau)$ converge to $Z_x(t)$ in the sense of finite-dimensional distributions.*

Proof. It is enough to prove the required convergence for $f \in C^2(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ only (by Theorem 2.1). Let such an f be chosen. Denote $f_k(x) = Ef(S_x^h(k))$. Then by the Markov property $f_k = R_h^k f$, where the operator R_h is defined via the formula

$$R_h f(x) = \int f(x + hy) p(x; dy).$$

Clearly each R_h is a positivity preserving contraction on $C_\infty(\mathbf{R}^d)$. On the other hand, the recurrent equation $f_k = R_h f_{k-1}$ can be rewritten as

$$\frac{f_k(x) - f_{k-1}(x)}{\tau} = h^{-\alpha} \int (f_{k-1}(x + hy) - f_{k-1}(x)) p(x; dy). \quad (8)$$

And this is a discrete time approximation to the equation

$$\frac{\partial f}{\partial t} = Lf \quad (9)$$

on the functions $f \in C^2(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ (and differentiable in t). Since this scheme is well-posed and stable (as it is solvable uniquely by the contraction R_h^n) and the solution to (9) is uniquely defined and preserves the space $C^2(\mathbf{R}^d) \cap C_\infty(\mathbf{R}^d)$ (by Theorem 2.1), it follows by the standard (and easy to prove) general results (see e.g. [20]) that the solutions to the finite-difference approximation converge to the solution of (9). Theorem is proved.

In case of p not depending on x , Theorem 2.3 turns to the known fact on the convergence of random walks with the distribution of jumps from the domain of normal attraction of a stable law to the corresponding stable Lévy motion.

3 Double-scaled random walks

To apply the developed theory to CTRW we shall need a generalization with multi-scaled walks that we present now.

We are interested in a process in $\mathbf{R}^d \times \mathbf{R}_+$ specified by the generator

$$\begin{aligned} \mathcal{L}f(x, u) &= \int_0^\infty \int_{S^{d-1}} (f(x + y, u) - f(x, u)) \frac{d|y|}{|y|^{1+\alpha}} S(x, u, \bar{y}) d_S \bar{y} \\ &\quad + \int_0^\infty (f(x, u + v) - f(x, u)) \frac{1}{v^{1+\beta}} w(x, u) dv. \end{aligned} \quad (10)$$

The following result (and its proof) is a straightforward generalization of Theorem 2.1.

Theorem 3.1 *Assume*

$$C_1 \leq \int_{S^{d-1}} |(\bar{p}, s)|^\alpha S(x, u, s) d_S s \leq C_2, \quad C_1 \leq w(x, u) \leq C_2$$

with some constants C_1, C_2 and that $S(x, s)$ and $w(x, u)$ have bounded derivatives with respect to x and u up to and inclusive order $q \geq 3$. Then the pseudo-differential operator (10) generates a Feller semigroup \mathcal{T}_t in $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ (continuous functions up to the boundary) with the space $(C^{q-1} \cap C_\infty)(\mathbf{R}^d \times \mathbf{R}_+)$ being its invariant core and hence a Feller process $(Y, V)(t)$ in $\mathbf{R}^d \times \mathbf{R}_+$.

We shall obtain now the corresponding extension of Theorems 2.2, 2.3.

Theorem 3.2 Assume $p(x, u; dydv)$ is a CFTP in $\mathbf{R}^d \times \mathbf{R}_+$, which is symmetric with respect to the reflection $y \mapsto -y$ and for which

$$p(x, u; \{0\} \times \mathbf{R}_+) + p(x, u; \mathbf{R}^d \times \{0\}) = 0.$$

Assume also that the projections belong to the domain of normal attraction of stable laws; more precisely, that uniformly in (x, u)

$$\int_{|y|>n} \int_{\bar{y} \in \Omega} p(x, u; dydv) \sim \frac{1}{\alpha n^\alpha} \int_{\Omega} S(x, u, s) d_S s, \quad n \rightarrow \infty, \quad (11)$$

and

$$\int_{v>n} \int_{|y|>A} p(x, u; dydv) \sim \frac{1}{\beta n^\beta} w(x, u, A), \quad n \rightarrow \infty, \quad (12)$$

for any $A \geq 0$ with a measurable function w of three arguments such that

$$w(x, u, 0) = w(x, u), \quad \lim_{A \rightarrow \infty} w(x, u, A) = 0 \quad (13)$$

(so that $w(x, u, A)$ is a measure on \mathbf{R}_+ for any x, u).

Consider the jump-type processes generated by

$$(\mathcal{L}_\tau f)(x, u) = \frac{1}{\tau} \int (f(x + \tau^{1/\alpha} y, u + \tau^{1/\beta} v)) - f(x, u) p(x, u; dydv). \quad (14)$$

Then the Feller semigroups \mathcal{T}_t^h in $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ of these processes (which are Feller, because \mathcal{L}_h is bounded in $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ for any h) converge to the semigroup \mathcal{T}_t .

Proof. As in Proposition 2.1 one deduces from (11), (12) that uniformly in x, u

$$\min(1, |y|^2) \int_0^\infty p(x, u; dy/h dv) h^{-\alpha} \rightarrow \min(1, |y|^2) \frac{d|y|}{|y|^{\alpha+1}} S(x, \bar{y}) d_S \bar{y}, \quad h \rightarrow 0, \quad (15)$$

and

$$\min(1, v) \int_{|y|>A} p(x, u; dydv/h) h^{-\beta} \rightarrow \min(1, v) w(x, u, A) \frac{dv}{v^{\beta+1}}, \quad h \rightarrow 0, \quad (16)$$

Next, assuming $f \in (C^2 \cap C_\infty)(\mathbf{R}^d \times \mathbf{R}_+)$ and writing

$$\mathcal{L}_\tau f(x, u) = I + II$$

with

$$I = \frac{1}{\tau} \int (f(x + \tau^{1/\alpha} y, u) - f(x, u)) p(x, u; dydv) + \frac{1}{\tau} \int (f(x, u + \tau^{1/\beta} v) - f(x, u)) p(x, u; dydv)$$

and

$$II = \frac{1}{\tau} \int [(f(x + \tau^{1/\alpha}y, u + \tau^{1/\beta}v) - f(x + \tau^{1/\alpha}y, u)) - (f(x, u + \tau^{1/\beta}v) - f(x, u))] p(x, u; dydv)$$

one observes that, as in the proof of Theorem 2.2, (15) and (16) (the latter with $A = 0$) imply that I converges to $\mathcal{L}f(x, u)$ uniformly in x, u . Thus in order to complete our proof we have to show that the function II converges to zero, as $\tau \rightarrow 0$. We have

$$II = \int (g(x + \tau^{1/\alpha}y, u, v) - g(x, u, v)) p(x, u; dydv / \tau^{1/\beta}) \frac{1}{\tau}$$

with

$$g(x, u, v) = f(x, u + v) - f(x, u).$$

By our assumptions on f

$$|g(x, u, v)| \leq C \min(1, v) (\max |\frac{\partial f}{\partial u}| + \max |f|) \leq \tilde{C} \min(1, v),$$

and

$$|\frac{\partial g}{\partial x}(x, u, v)| \leq C \min(1, v) (\max |\frac{\partial^2 f}{\partial u \partial x}| + \max |\frac{\partial f}{\partial x}|) \leq \tilde{C} \min(1, v)$$

with some constants C and \tilde{C} . Hence by (16) and (13) for an arbitrary $\epsilon > 0$ there exists a A such that

$$\int_{|y| > A} (g(x + \tau^{1/\alpha}y, u, v) - g(x, u, v)) p(x, u; dydv / \tau^{1/\beta}) \frac{1}{\tau} < \epsilon;$$

and on the other hand, for an arbitrary A

$$\int_{|y| < A} (g(x + \tau^{1/\alpha}y, u, v) - g(x, u, v)) p(x, u; dydv / \tau^{1/\beta}) \frac{1}{\tau} \leq \tau^{1/\alpha} A \kappa$$

with a constant κ so that II can be made arbitrary small by first choosing large enough A and then choosing small enough τ .

Define now the process $(Y, V)_{x,u}^\tau(t/\tau) = (Y, V)_{x,u}^\tau([t/\tau])$, where

$$(Y, V)_{x,u}^\tau(0) = (x, u), \quad (Y, V)_{x,u}^\tau(1) = (x + \tau^{1/\alpha}Y_1, u + \tau^{1/\beta}V_1), \dots,$$

$$(Y, V)_{x,u}^\tau(j) = (Y, V)_{x,u}^\tau(j-1) + (\tau^{1/\alpha}Y_j, \tau^{1/\beta}V_j), \dots$$

and each pair (Y_j, V_j) is distributed according to $p((Y, V)_{x,u}^\tau(j-1); dydv)$. If $p(x, u; dydv)$ does not depend on x, u , then

$$(Y, V)_{x,u}^\tau(n) = (x, u) + (\tau^{1/\alpha}(Y_1 + \dots + Y_n), \tau^{1/\beta}(V_1 + \dots + V_n)).$$

In view of Theorem 3.2 the following result is obtained by literally the same arguments as Theorem 2.3.

Theorem 3.3 *Under the assumptions of Theorems 3.1 and 3.2 the linear contractions $Ef((Y, V)_{x,u}^\tau(t/\tau))$ in $C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$ converge to the semigroup $\mathcal{T}_t f(x, u)$ of the process $(Y, V)(t)$ uniformly on $t \in [0, t_0]$, as $\tau \rightarrow 0$.*

4 Subordination by hitting times and generalized fractional evolutions

Let $X(u)$, $u \geq 0$ be a Lévy subordinator, i.e. an increasing i.i.d. càdlàg Feller process (adapted to a filtration on a suitable probability space) with the generator

$$Af(x) = \int_0^\infty (f(x+y) - f(x))\nu(dy) + a \frac{\partial f}{\partial x}, \quad (17)$$

where $a \geq 0$ and ν is a Borel measure on $\{y > 0\}$ such that

$$\int_0^\infty \min(1, y)\nu(dy) < \infty.$$

We are interested in the inverse function process or the first hitting time process $Z(t)$ defined as

$$Z_X(t) = Z(t) = \inf\{u : X(u) > t\} = \sup\{u : X(u) \leq t\}, \quad (18)$$

which is of course also an increasing càdlàg process. To make our further analysis more transparent (avoiding heavy technicalities of the most general case) we shall assume that there exist $\epsilon > 0$ and $\beta \in (0, 1)$ such that

$$\nu(dy) \geq y^{1+\beta}, \quad 0 < y < \epsilon. \quad (19)$$

For convenient reference we collect in the next statement (without proofs) the elementary (well known) properties of $X(u)$.

Proposition 4.1 *Under condition (19) (i) the process $X(u)$ is a.s. increasing at each point, i.e. it is not a constant on any finite time interval; (ii) distribution of $X(u)$ for $u > 0$ has a density $G(u, y)$ vanishing for $y < 0$, which is infinitely differentiable in both variable and satisfies the equation*

$$\frac{\partial G}{\partial u} = A^*G, \quad (20)$$

where A^* is the dual operator to A given by

$$A^*f(x) = \int_0^\infty (f(x-y) - f(x))\nu(dy) - a \frac{\partial f}{\partial x},$$

(iii) if extended by zero to the half-space $\{t < 0\}$ the locally integrable function $G(t, y)$ on \mathbf{R}^2 specifies a generalized function satisfying (in the sense of distribution) the equation

$$\frac{\partial G}{\partial u} = A^*G + \delta(u)\delta(y). \quad (21)$$

Corollary 1 *Under condition (19) (i) the process $Z(t)$ is a.s. continuous and $Z(0) = 0$; (ii) the distribution of $Z(t)$ has a continuously differentiable probability density function $Q(t, u)$ for $u > 0$ given by*

$$Q(t, u) = -\frac{\partial}{\partial u} \int_{-\infty}^t G(u, y) dy. \quad (22)$$

Proof. (i) follows from Proposition 4.1 (i) and for (ii) one observes that

$$P(Z(t) \leq u) = P(X(u) \geq t) = \int_t^\infty G(u, y) dy = 1 - \int_0^t G(u, y) dy$$

which implies (22) by the differentiability of G .

Theorem 4.1 *Under condition (19) the density Q satisfies the equation*

$$A^*Q = \frac{\partial Q}{\partial u} \quad (23)$$

for $u > 0$, where A^* acts on the variable t , and the boundary condition

$$\lim_{u \rightarrow 0} Q(t, u) = -A^*\theta(t) \quad (24)$$

where $\theta(t)$ is the indicator function equal one (respectively 0) for positive (respectively negative) t . If Q is extended by zero to the half-space $\{u < 0\}$, it satisfies the equation

$$A^*Q = \frac{\partial Q}{\partial u} + \delta(u)A^*\theta(t), \quad (25)$$

in the sense of distribution (generalized functions).

Moreover the (point-wise) derivative $\frac{\partial Q}{\partial t}$ also satisfies equation (23) for $u > 0$ and satisfies the equation

$$A^*\frac{\partial Q}{\partial t} = \frac{\partial}{\partial u} \frac{\partial Q}{\partial t} + \delta(u) \frac{d}{dt} A^*\theta(t) \quad (26)$$

in the sense of distributions.

Remark 5 *In the case of a β -stable subordinator $X(u)$ with the generator*

$$Af(x) = -\frac{1}{\Gamma(-\beta)} \int_0^\infty (f(x+y) - f(x))y^{-1-\beta} dy, \quad (27)$$

one has

$$A = -\frac{d^\beta}{d(-t)^\beta}, \quad A^* = -\frac{d^\beta}{dt^\beta} \quad (28)$$

(these equations can be considered as the definitions of fractional derivatives; we refer to books [16] and [18] for a general background in fractional calculus; a short handy account is given also in Appendix to [19]), in which case equation (25) takes the form

$$\frac{d^\beta Q}{dt^\beta} + \frac{\partial Q}{\partial u} = \delta(u) \frac{t^{-\beta}}{\Gamma(1-\beta)} \quad (29)$$

coinciding with (B14) from [19].

Proof. Notice that by (22), (20) and by the commutativity of the integration and A^* one has

$$Q(t, u) = - \int_{-\infty}^t \frac{\partial}{\partial u} G(u, y) dy = - \int_{-\infty}^t (A^*G(u, \cdot))(y) dy = -A^* \int_{-\infty}^t G(u, y) dy.$$

This implies (23) (by differentiating with respect to u and again using (22)) and (24), because $G(0, y) = \delta(y)$.

Assume now that Q is extended by zero to $\{u < 0\}$. Let ϕ be a test function (infinitely differentiable with a compact support) in \mathbf{R}^2 . Then in the sense of distribution

$$\begin{aligned} \left(\left(\frac{\partial}{\partial u} - A^* \right) Q, \phi \right) &= \left(Q, \left(-\frac{\partial}{\partial u} - A \right) \phi \right) \\ &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} du \int_{\mathbf{R}} dt Q(t, u) \left(-\frac{\partial}{\partial u} - A \right) \phi(t, u) \\ &= \lim_{\epsilon \rightarrow 0} \left[\int_{\epsilon}^{\infty} du \int_{\mathbf{R}} dt \phi(t, u) \left(\frac{\partial}{\partial u} - A^* \right) Q(t, u) + \int_{\mathbf{R}} \phi(t, \epsilon) Q(t, \epsilon) dt \right]. \end{aligned}$$

The first term here vanishes by (23). Hence by (24)

$$\left(\left(\frac{\partial}{\partial u} - A^* \right) Q, \phi \right) = - \int_{\mathbf{R}} \phi(t, 0) A^* \theta(t) dt,$$

which clearly implies (25). The required properties of $\frac{\partial Q}{\partial t}$ follows similarly from the representation

$$\frac{\partial Q}{\partial t}(t, u) = -\frac{\partial G}{\partial u}(u, t).$$

We are interested now in the random time change of Markov processes specified by the process $Z(t)$.

Theorem 4.2 *Under the conditions of Theorem 4.1 let $Y(t)$ be a Feller process in \mathbf{R}^d , independent of $Z(t)$, and with the domain of the generator L containing $(C_{\infty} \cap C^2)(\mathbf{R}^d)$. Denote the transition probabilities of $Y(t)$ by*

$$T(t, x, dy) = P(Y_x(t) \in dy) = P_x(Y(t)).$$

Then the distributions of the (time changed or subordinated) process $Y(Z(t))$ for $t > 0$ are given by

$$P_x(Y(Z(t)) \in dy) = \int_0^{\infty} T(u, x, dy) Q(t, u) du, \quad (30)$$

the averages $f(t, x) = Ef(Y_x(Z(t)))$ of $f \in (C_{\infty} \cap C^2)(\mathbf{R}^d)$ satisfy the (generalized) fractional evolution equation

$$A_t^* f(t, x) = -L_x f(t, x) + f(x) A^* \theta(t) \quad (31)$$

(where the subscripts indicate the variables, on which the operators act), and their time derivatives $h = \partial f / \partial t$ satisfy for $t > 0$ the equation

$$A_t^* h = -L_x h + f(x) \frac{d}{dt} A^* \theta(t). \quad (32)$$

Moreover, if $Y(t)$ has a smooth transition probability density so that $T(t, x, dy) = T(t, x, y) dy$ and the forward and backward equations

$$\frac{\partial T}{\partial t}(t, x, y) = L_x T(t, x, y) = L_y^* T(t, x, y) \quad (33)$$

hold, then the distributions of $Y(Z(t))$ have smooth density

$$g(t, x, y) = \int_0^\infty T(u, x, y) Q(t, u) du \quad (34)$$

satisfying the forward (generalized) fractional evolution equation

$$A^*g = -L_y^*g + \delta(x - y)A^*\theta(t) \quad (35)$$

and the backward (generalized) fractional evolution equation

$$A^*g = -L_xg + \delta(x - y)A^*\theta(t) \quad (36)$$

with the time derivative $h = \partial g / \partial t$ satisfying for $t > 0$ the equation

$$A^*h = -L_y^*h + \delta(x - y)\frac{d}{dt}A^*\theta(t) \quad (37)$$

Remark 6 In the case of a β -stable Lévy subordinator $X(u)$ with the generator (27), where (28) hold, the left hand sides of the above equations become fractional derivatives per se. In particular, if $Y(t)$ is a symmetric α -stable Lévy motion, equation (35) takes the form

$$\frac{\partial^\beta}{\partial t^\beta} g(t, y - x) = \frac{\partial^\alpha}{\partial |y|^\alpha} g(t, y - x) + \delta(y - x) \frac{t^{-\beta}}{\Gamma(1 - \beta)}, \quad (38)$$

deduced in [19] and [21]. The corresponding particular case of (34) also appears in [14] as well as in [19], where it is called a formula of separation of variables. Our general approach makes it clear that this separation of variables comes from the independence of $Y(t)$ and the subordinator $X(u)$ (see Proposition 4.4 for a more general situation).

Proof. For a continuous bounded function f one has for $t > 0$ that

$$Ef(Y_x(Z(t))) = \int_0^\infty E(f(Y_x(Z(t)) | Z(t) = u) Q(t, u) du = \int_0^\infty Ef(Y_x(u)) Q(t, u) du$$

by the independence of Z and Y . This implies (30) and (34).

From Theorem 4.1 it follows that for $t > 0$

$$\begin{aligned} A_t^*g &= \lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty G(u, x, y) A_t^*Q(t, u) du = \lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty G(u, x, y) \frac{\partial}{\partial u} Q(t, u) du \\ &= - \int_0^\infty \frac{\partial}{\partial u} G(u, x, y) Q(t, u) du + \delta(x - y) A^*\theta(t), \end{aligned}$$

where by (33) the first term equals $-L_y^*g = L_xg$, implying (35) and (36). Other equations are proved analogously.

Now we like to generalize this theory to the case of Lévy type subordinators $X(u)$ specified by the generators of the form

$$Af(x) = \int_0^\infty (f(x + y) - f(x)) \nu(x, dy) + a(x) \frac{\partial f}{\partial x} \quad (39)$$

with position depending Lévy measure and drift. We need some regularity assumptions in order to have a smooth transition probability density like in case of the Lévy motions.

Proposition 4.2 Assume that (i) ν has a density $\nu(x, y)$ with respect to Lebesgue measure such that

$$C_1 \min(y^{-1-\beta_1}, y^{-1-\beta_2}) \leq \nu(x, y) \leq C_2 \max(y^{-1-\beta_1}, y^{-1-\beta_2}) \quad (40)$$

with some constants $C_1, C_2 > 0$ and $0 < \beta_1 < \beta_2 < 1$ (ii) ν is thrice continuously differentiable with respect to x with the derivatives satisfying the same estimate (40), (iii) $a(x)$ is non-negative with bounded derivatives up to the order three. Then the generator (39) specifies an increasing Feller process having for $u > 0$ a transition probability density $G(u, y) = P(X(u) \in dy)$ (we assume that $X(u)$ starts at the origin) that is twice continuously differentiable in u .

Remark 7 Condition (40) holds for popular stable-like processes with a position dependent stability index.

Proof. The existence of the Feller process is proved under much more general assumptions in [1]. A proof of the existence of a smooth transition probability density is given in [8] under slightly different assumptions (symmetric multidimensional stable-like processes), but is easily seen to be valid in the present situation.

One can see now that the hitting time process defined by (18) with $X(u)$ from the previous Proposition is again continuous and has a continuously differentiable density $Q(t, u)$ for $t > 0$ given by (22). However (23) does not hold, because the operators A and integration do not commute. On the other hand, equation (26) remains true (as easily seen from the proof). This leads directly to the following partial generalization of Theorem 4.2.

Proposition 4.3 Let $Y(t)$ be the same Feller process in \mathbf{R}^d as in Theorem 4.3, but independent hitting time process $Z(t)$ be constructed from $X(u)$ under the assumptions of Proposition 4.2.

Then the distributions of the (time changed or subordinated) process $Y(Z(t))$ for $t > 0$ are given by (30) and the time derivatives $h = \partial f / \partial t$ of the averages $f(t, x) = Ef(Y_x(Z(t)))$ of continuous bounded functions f satisfy (37).

At last we like to extend this to the case of dependent hitting times.

Proposition 4.4 Let $(Y, V)(t)$ be a random process in $\mathbf{R}^d \times \mathbf{R}_+$ such that (i) the components $Y(t), V(s)$ at different times have a joint probability density

$$\phi(s, u; y, v) = P(Y(s) \in dy, V(u) \in dv)$$

that is continuously differentiable in u for $u, s > 0$, and (ii) the component $V(t)$ is increasing and is a.s. not a constant on any finite interval. For instance, the process from Theorem 3.1 enjoys these properties. Then (i) the hitting time process $Z(t) = Z_V(t)$ (defined by (18) with V instead of X) is a.s. continuous, (ii) there exists a continuous joint probability density of $Y(s), Z(t)$ given by

$$g_{Y(s), Z(t)}(y, u) = \frac{\partial}{\partial u} \int_t^\infty \phi(s, u; y, v) dv \quad (41)$$

and (iii) the distribution of the composition $Y(Z(t))$ has the probability density

$$\Phi_{Y(Z(t))}(y) = \int_0^\infty g_{Y(s), Z(t)}(y, s) ds = \int_0^\infty \left(\frac{\partial}{\partial u} \int_t^\infty \phi(s, u; y, v) dv \right) |_{u=s} ds. \quad (42)$$

Proof. (i) and (ii) are straightforward extensions of the Corollary to Proposition 4.1. Statement (iii) follows from conditioning and the definition of the joint distribution.

5 Limit theorems for position dependent CTRW

Now everything is ready for our main result.

Theorem 5.1 *Under the assumptions of Theorems 3.1 and 3.2 let $Z^\tau(t), Z(t)$ be the hitting time processes for $V^\tau(t/\tau)$ and $V(t)$ respectively (defined by the corresponding formula (18)). Then the subordinated processes $Y^\tau(Z^\tau(t)/\tau)$ converge to the subordinated process $Y(Z(t))$ in the sense of marginal distributions, i.e.*

$$E_{x,0}f(Y^\tau(Z^\tau(t)/\tau)) \rightarrow E_{x,0}(Y(Z(t))), \quad \tau \rightarrow 0, \quad (43)$$

for arbitrary $x \in \mathbf{R}^d$, $f \in C_\infty(\mathbf{R}^d \times \mathbf{R}_+)$, uniformly for t from any compact interval.

Remark 8 *We show the convergence in the weakest possible sense. It does not seem difficult to extend it to the convergence in the Skorokhod space of trajectories using standard tools (compactness etc) or the theory of continuous compositions from [22]. Similar result holds for the continuous time approximation from Theorem 3.2.*

Proof. Since the time is effectively discrete in $V^\tau(t/\tau)$, it follows that

$$Z^\tau(t) = \max\{u : X(u) \leq t\},$$

and that the events $(Z^\tau(t) \leq u)$ and $(V^\tau(u/\tau) \geq t)$ coincide, which implies that the convergence of finite dimensional distributions of $(Y^\tau(s/\tau), V^\tau(u/\tau))$ to $(Y(s), V(u))$ (proved in Theorem 3.3) is equivalent to the corresponding convergence of the distributions of $(Y^\tau(s/\tau), Z^\tau(t))$ to $(Y(s), Z(t))$.

Next, since $V(0) = 0$, is continuous and $V(u) \rightarrow \infty$ as $u \rightarrow \infty$ and because the limiting distribution is absolutely continuous, to show (43) it is sufficient to show that

$$P_{x,0}[Y^\tau(Z_K^\tau(t)/\tau) \in A] \rightarrow P_{x,0}[Y(Z_K(t)) \in A], \quad \tau \rightarrow 0, \quad (44)$$

for large enough $K > 0$ and any compact set A , whose boundary has Lebesgue measure zero, where

$$Z_K^\tau(t) = Z^\tau(t), \quad K^{-1} \leq Z^\tau(t) \leq K,$$

and vanishes otherwise, and similarly $Z_K(t)$ is defined.

Now

$$P[Y^\tau(Z_K^\tau(t)/\tau) \in A] = \sum_{k=1/K\tau}^{K/\tau} P[V^\tau(k) \in A \text{ \& } Z^\tau(t) \in [k\tau, (k+1)\tau]] \quad (45)$$

and

$$P[Y(Z_K(t)) \in A] = \sum_{k=1/K\tau}^{K/\tau} \int_A dy \int_{\tau k}^{\tau(k+1)} g_{Y(s), Z(t)}(y, s) ds, \quad (46)$$

which can be rewritten as

$$\sum_{k=1/K\tau}^{K/\tau} \int_A dy \int_{\tau k}^{\tau(k+1)} g_{Y(\tau k), Z(t)}(y, s) ds + \sum_{k=1/K\tau}^{K/\tau} \int_A dy \int_{\tau k}^{\tau(k+1)} (g_{Y(s), Z(t)} - g_{Y(\tau k), Z(t)})(y, s) ds. \quad (47)$$

The second term here tends to zero as $\tau \rightarrow 0$ due to the continuity of the function (41), and the difference between the first term and (45) tends to zero, because the distributions of $(Y^\tau(s/\tau), Z^\tau(t))$ converge to the distribution of $(Y(s), Z(t))$. Hence (44) follows. Theorem is proved.

In the case when S does not depend on u and w does not depend on x in (10), the limiting process $(Y, V)(t)$ has independent components so that the averages of the limiting subordinated process satisfy the generalized fractional evolution equation from Proposition 4.3, and if moreover w is a constant, they satisfy the fractional equations from Theorem 4.2. In particular, if $p(x, u, dydv)$ does not depend on (x, u) and decomposes into a product $p(dy)q(dv)$, and the limit $V(t)$ is stable, we recover the main result from [14] (in a slightly less general setting, since we worked with symmetric stable laws and not with operator stable motions as in [14]), as well as of course the corresponding results from [7], [11] (put $t = 1$ in (43)) on the long time behavior of the normalized subordinated sums (1).

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References

- [1] R.F. Bass. Uniqueness in law for pure jump type Markov processes. *Probab. Theory Related Fields* **79** (1988), 271-287.
- [2] V. Yu. Korolev, V. E. Bening, S.Ya. Shorgin. *Matematicheskie osnovy teorii riska* (in Russian), Moscow, Fismatlit, 2007.
- [3] V. Bening, V. Korolev, T. Suchorukova, G. Gusarov, V. Saenko, V. Kolokoltsov. Fractionally Stable Distributions. In: V. Korolev, N. Skvortsova (Eds.) "Stochastic Models of Plasma Turbulence", Moscow State University, Moscow, 2003, p. 291-360 (in Russian). Engl. transl. in V. Korolev, N. Skvortsova (Eds.) "Stochastic Models of Structural Plasma Turbulence", VSP, Boston 2006, p.175-244.
- [4] V.E. Bening, V.Yu. Korolev, V.N. Kolokoltsov, V.V. Saenko, V.V. Uchaikin, V.M. Zolotarev. Estimation of parameters of fractional stable distributions. *J. Math. Sci. (N. Y.)* **123:1** (2004), 3722–3732.
- [5] V.E. Bening, V. Yu. Korolev, V.N. Kolokoltsov. Limit theorems for continuous-time random walks in the double array limit scheme. *J. Math. Sci. (N.Y.)* **138:1** (2006), 5348-5365.
- [6] O. Kallenberg. *Foundations of Modern Probability*. Second ed., Springer 2002.
- [7] M. Kotulski. Asymptotic Distribution of Continuous-Time Random Walks: a Probabilistic Approach. *J. Stat. Phys.* **81:3/4** (1995), 777-792.
- [8] V. N. Kolokoltsov. Symmetric Stable Laws and Stable-Like Jump-Diffusions. *Proc. London Math. Soc.* **3:80** (2000), 725-768.

- [9] V. N. Kolokoltsov. Semiclassical Analysis for Diffusions and Stochastic Processes. Springer Lecture Notes in Math. v. 1724, 2000.
- [10] V. Kolokoltsov. Nonlinear Markov Semigroups and Interacting Lévy Type Processes. Journ. Stat. Physics **126:3** (2007), 585-642.
- [11] V. Kolokoltsov, V. Korolev, V. Uchaikin. *Fractional Stable Distributions*. J. Math. Sci. (N.Y.) **105:6** (2001), 2570-2577.
- [12] V. Korolev et al. Some methods of the analysis of time characteristics of catastrophes in non-homogeneous flows of extremal events. In: I.A. Sokolov (Ed.) *Sistemi i sredstva informatiki. Matematicheskie modeli v informacionnich technologiach*. Moscow, RAN, 2006 (In Russian), p. 5-23.
- [13] V.P. Maslov. Perturbation Theory and Asymptotical Methods. Moscow State University Press, 1965 (in Russian). French Transl. Dunod, Paris, 1972.
- [14] M.M. Meerschaert, H.-P. Scheffler. Limit Theorems for Continuous-Time Random Walks with Infinite Mean Waiting Times. J. Appl. Prob. **41** (2004), 623-638.
- [15] R. Metzler, J. Klafter. The Random Walk's Guide to Anomalous Diffusion: A Fractional Dynamic Approach. Phys. Rep. **339** (2000), 1-77.
- [16] K. S. Miller, B. Ross. An Introduction to the Fractional Differential Equations. Wiley, New York, 1993.
- [17] E.W. Montroll, G.H. Weiss. Random Walks on Lattices, II. J. Math. Phys. **6** (1965), 167-181.
- [18] A.I. Saichev, W.A. Woyczynski. Distributions in the Physical and Engineering Sciences. Birkhäuser, Boston, 1997, v. 1.
- [19] A.I. Saichev, G.M. Zaslavsky. Fractional kinetic equations: solutions and applications. Chaos **7:4** (1997), 753-764.
- [20] A.A. Samarskii. Teoriya raznostnykh skhem (Russian) [Theory of difference schemes]. Third ed. "Nauka", Moscow, 1989.
- [21] V.V. Uchaikin. Montroll-Weisse Problem, Fractional Equations and Stable Distributions. Intern. Journ. Theor. Phys. **39:8** (2000), 2087-2105.
- [22] W. Whit. Stochastic-Processes Limits. Springer 2002.
- [23] G.M. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. Pgysica D **76** (1994), 110-122.